Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 5, pp. 761–770, 1993

0021–8928/93 $24.00+0.00

0021-8928(93)E0022-O

THE MINIMAX PRINCIPLE IN MECHANICS†

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(*Received* 7 *July* 1992)

The motion of the holonomic mechanical systems considered here takes place in a potential force field and is described by Lagrange equations of the second kind. It is shown that the solution of any problem involving these equations, whatever the conditions, is an extremal of a certain functional whose structure depends exclusively on the Lagrangian and on the specified conditions.

LAGRANGE'S principle gives a variational description of the solutions of a two-point boundary-value problem for the Lagrange equations, given the generalized coordinates of the system at the initial and final instants of time. No such variational description is available for the solutions of other problems involving the Lagrange equations. In particular, we do not know whether, say, the solutions of a Cauchy problem for the Lagrange equations are extremals of some functional. This explains the interest in variational principles that might enable one to single out solutions of Lagrange's equations satisfying arbitrary boundary, initial and intermediate conditions and conditions of the inclusion type. Having such principles at our disposal, one could envisage the use of variational methods to solve a variety of problems, including Cauchy problems for Lagrange's equations. It might thus be possible to solve problems not by integrating the equations of motion but by determining a suitable set of extremals.

1. STATEMENT OF THE PROBLEM

Let us consider a holonomic mechanical system with n degrees of freedom in which motion takes place in a potential force field. In independent generalized coordinates q_i (here and everywhere henceforth $i=1, 2, \ldots, n$) the motion of the system may be described by a system of Lagrange differential equations of the second kind

$$\frac{d}{dt}\frac{\partial L(q,q^{\cdot},t)}{\partial q_i^{\cdot}} - \frac{\partial L(q,q^{\cdot},t)}{\partial q_i^{\cdot}} = 0 \tag{1.1}$$

Equations (1.1) also govern processes in electrical and electromechanical systems [1] subject to some idealization.

It is assumed that the Lagrangian $L(q, q^{\cdot}, t)$ is a strongly convex function of the generalized velocities $q^{\cdot}$, i.e. for any q, $q^{\cdot}$, ξ, t one has the inequality

$$\sum_{i,j=1}^{n} \frac{\partial^2 L(q,q^{\cdot},t)}{\partial q_i^{\cdot}\,\partial q_j^{\cdot}}\xi_i\xi_j \geqslant \alpha\sum_{i=1}^{n}\xi_i^2, \quad \alpha>0 \tag{1.2}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 5, pp. 3–13, 1993.

In natural systems [1], inequality (1.2) is a direct consequence of the representation $L(q, q^{\cdot}, t) = T(q, q^{\cdot}, t) - \Pi(q, t)$, since the kinetic energy $T(q, q^{\cdot}, t)$ is a polynomial of the second degree in $q_i^{\cdot}$

$$T = T_2 + T_1 + T_0, \quad T_1 = \sum_{k=1}^{n} a_k(q,t)\, q_k^{\cdot}, \quad T_0 = T_0(q,t)$$

$$T_2 = \sum_{i,k=1}^{n} \frac{1}{2} a_{ik}(q,t)\, q_i^{\cdot} q_k^{\cdot}$$

where T_2 is a positive definite quadratic form in the generalized velocities $q_i^{\cdot}$.

In some problems condition (1.2) may hold for only certain values of the generalized velocities. For example, if one is considering a relativistic particle moving when there is no field [2], which is not a natural system, then the corresponding Lagrangian

$$L = -mc^2\, [1 - (x_1^{\cdot 2} + x_2^{\cdot 2} + x_3^{\cdot 2})/c^2]^{1/2}$$

will be convex in the domain $v^2 = (x_1^{\cdot 2} + x_2^{\cdot 2} + x_3^{\cdot 2}) < c^2$. To enable us to consider such cases, we shall formulate all our results in such a way that, suitably generalized, they will carry over to cases in which inequality (1.2) is true not for all $q^{\cdot}$ but only for those in some bounded domain.

Lagrange's equations (1.1) form a system of second-order ordinary differential equations

$$\sum_{j=1}^{n} \left(\frac{\partial^2 L}{\partial q_j^{\cdot} \partial q_i^{\cdot}} q_j^{\cdot\cdot} + \frac{\partial^2 L}{\partial q_j \partial q_i^{\cdot}} q_j^{\cdot} \right) + \frac{\partial^2 L}{\partial t \partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = 0 \tag{1.3}$$

Condition (1.2) implies that the Hessian of the Lagrangian does not vanish ($\det \| \partial^2 L / \partial q_i^{\cdot} \partial q_j^{\cdot} \|_{i,j=1}^{n} \neq 0$), so Eqs (1.3) can be reduced to Cauchy normal form

$$q_i^{\cdot\cdot} = f_i(q, q^{\cdot}, t) \tag{1.4}$$

The solutions of a two-point problem for Lagrange's equations admit of a variational description through the use of Hamilton's principle [1]† restricted to trajectories of system (1.1) joining points (a, t_0) and (b, t_1) of $(n+1)$-dimensional space $\{q, t\}$

$$q_i^0 = q_i(t_0) = a_i; \quad q_i^1 = q_i(t_1) = b_i; \quad t_1 > t_0 \tag{1.5}$$

the Hamilton action.

$$W[q(t)] = \int_{t_0}^{t_1} L(q(t), q^{\cdot}(t), t)\, dt, \quad q(t) = \| q_i(t) \|_{i=1}^{n} \tag{1.6}$$

has a stationary value relative to its action on any other C^1 curves through the same two points.

It is essential for the proof of this principle that all admissible curves pass through the points (a, t_0) and (b, t_1). At the same time, it would be important to have variational principles that determine solutions of system (1.1) with arbitrary boundary and intermediate conditions. Using such principles one could obtain, say, a variational description of the solutions of Cauchy's problem for system (1.1), a problem with periodic solutions, etc. For example, if a variational description of solutions of Cauchy's problem for the equations of motion (1.1) were

†See also RUMYANTSEV V. V., On the fundamental laws and variational principles of classical mechanics. Preprint No. 257, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1985.

available, the use of direct methods of the calculus of variations, would enable one to replace the problem of integrating Eqs (1.1) by the problem of determining appropriate extremals.

The purpose of this paper is to show that any solution $q(t)$ of Lagrange's equations (1.1) satisfying conditions of the form

$$F_j[q(t)] \leq 0 \quad (j=1,2,\ldots,k) \tag{1.7}$$

where F_j, generally speaking, are non-linear functionals, is at the same time the solution of a certain variational problem. Arbitrary conditions may be represented in this form, including the initial conditions, boundary conditions, and point conditions of the general form

$$h_\nu(q(\tau), q^{\cdot}(\tau)) = 0, \quad t_0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m \leq t_1 \quad (\nu=1,2,\ldots,r) \tag{1.8}$$

Conditions (1.7) also include the conditions

$$\chi_p(q(t), q^{\cdot}(t)) \leq 0 \quad (p=1,2,\ldots,s); \quad \{q(t),\ q^{\cdot}(t)\} \in D(t) \tag{1.9}$$

defining a set of inequalities and inclusions, where $D(t)$ denotes a time-dependent domain in the phase space $\{q,\ q^{\cdot}\}$.

2. THE VARIATIONAL PRINCIPLE WHEN THERE ARE NO CONJUGATE POINTS

In order to explain the basic idea of the variational principle to be established here, we will first consider a boundary-value problem for Eqs (1.1) with boundary conditions (1.5), which has a unique solution for any a, b, $t_1 > t_0$. This condition means that no interval contains conjugate points (in Jacobi's sense—conjugate kinetic foci) [1, 3] (see also the paper mentioned in the footnote above). For such systems (1.1) the Hamilton action (1.6) restricted to a solution $q(t)$ of problem (1.1), (1.5) reaches its least value relative to the action on other curves between points $(a,\ t_0)$ and $(b,\ t_1)$ of the extended coordinate space $\{q, t\}$.

With every curve $q(t)$ of class $C^1[t_0,\ t_1]$ we associate a set $I\{q\} = I\{q(t)\}$ of functions $z(t) = \| z_i(t) \|_{i=1}^n$

$$I\{q\} = \{z(t) \in C^1[t_0, t_1]:\ z(t_0) = q(t_0),\ \ z(t_1) = q(t_1)\} \tag{2.1}$$

The set $I\{q\}$ is a pencil of curves $z(t)$ of class C^1 passing through points $\{q(t_0),\ t_0\}$ and $\{q(t_1),\ t_1\}$ of the extended coordinate space $\{q,\ t\}$. Obviously, the initial curve $q(t) \in I\{q(t)\}$ is also an element of this pencil.

Consider the functional

$$\theta[q(t), z(t)] = \int_{t_0}^{t_1} \{L(q(t), q^{\cdot}(t), t) - L(z(t), z^{\cdot}(t), t)\}\, dt \tag{2.2}$$

which is defined for functions $q(t) \in C^1[t_0,\ t_1]$, $z(t) \in I\{q\}$. Using the functional $\theta[q(t), z(t)]$, and Hamilton's principle, we define a non-negative functional

$$\begin{aligned} V[q(t)] &= \theta[q(t), z_q(t)] = \\ &= \max\left\{\int_{t_0}^{t_1} \{L(q, q^{\cdot}, t) - L(z, z^{\cdot}, t)\}\, dt \,\Big|\, z(t) \in I\{q\}\right\} \end{aligned} \tag{2.3}$$

The existence of the maximum on the right of (2.3) is due to the absence of conjugate points, since we have assumed that the boundary-value $z_q(t) \in I\{q\}$ for system (1.1), written here in terms of the variables z_i

$$\frac{d}{dt}\frac{\partial L(z,z^{\cdot},t)}{\partial z_i^{\cdot}} - \frac{\partial L(z,z^{\cdot},t)}{\partial z_i} = 0, \quad z(t) \in I\{q\} \tag{2.4}$$

has a unique solution. The condition $z(t) \in I\{q\}$, together with the condition $z(t) \in C^1[t_0, t_1]$, implies that $z(t_0) = q(t_0)$, $z(t_1) = q(t_1)$.

If the system has no conjugate points, a principal Hamiltonian function $W[b, t_1, a, t_0]$ exists expressing the value of the functional (1.6) in terms of $a = q^0$, $b = q^1$, t_0, t_1 on solutions of system (2.4) subject to conditions (1.5) [1]. Put $W[b, t_1, a, t_0] = W[q^1, t_1, q^0, t_0]$. As we know [1], the principal Hamiltonian function $W[q, t, q^0, t_0]$ satisfies the Hamilton–Jacobi equation

$$\partial W/\partial t + H(q, \partial W/\partial q, t) = 0 \tag{2.5}$$

where, by virtue of (1.2), the Hamiltonian is defined by [4]

$$H(q,p,t) = \max_{\xi \in R^n}\left[\sum_{i=1}^n p_i\xi_i - L(q,\xi,t)\right] \tag{2.6}$$

For any function $\varphi(t) \in I\{q\}$ (i.e. for $\varphi(t_0) = q^0$, $\varphi(t_1) = q^1$), we have

$$\int_{t_0}^{t_1} \frac{dW[\varphi, t, q^0, t_0]}{dt}dt = W[\varphi(t_1), t_1, q^0, t_0] - W[\varphi(t_0), t_0, q^0, t_0] =$$

$$= W[q^1, t_1, q^0, t_0] - W[q^0, t_0, q^0, t_0] = W[q^1, t_1, q^0, t_0]$$

since $W[q^0, t_0, q^0, t_0] = 0$. Let $z_q(t) \in I\{q\}$ denote the solution $z(t)$ of system (2.4) satisfying conditions (2.1). Then

$$W[z_q(t)] = \int_{t_0}^{t_1} L(z_q, z_q^{\cdot}, t)dt = \int_{t_0}^{t_1} \frac{dW[z_q, t, q^0, t_0]}{dt}dt = \int_{t_0}^{t_1} \frac{dW[\varphi, t, q^0, t_0]}{dt}dt$$

for $\varphi(t) \in I\{q\}$. Taking these relationships into account, as well as (1.6), (2.5) and (2.6), we see that for any functions $\varphi(t) \in I\{q\}$

$$\Delta W = W[\varphi(t)] - W[z_q(t)] = \int_{t_0}^{t_1} \{L(\varphi(t), \varphi^{\cdot}(t), t) - L(z_q(t), z_q^{\cdot}(t), t)\}\, dt =$$

$$= \int_{t_0}^{t_1}\left\{L(\varphi, \varphi^{\cdot}, t) - \frac{dW[\varphi, t, q^0, t_0]}{dt}\right\}dt =$$

$$= \int_{t_0}^{t_1}\left\{L(\varphi, \varphi^{\cdot}, t) - \frac{\partial W[\varphi, t, q^0, t_0]}{\partial t} - \sum_{i=1}^n \frac{\partial W}{\partial \varphi_i}\varphi_i^{\cdot}\right\}dt =$$

$$= \int_{t_0}^{t_1}\left\{H\left(\varphi, \frac{\partial W}{\partial \varphi}, t\right) - \left[\sum_{i=1}^n \frac{\partial W}{\partial \varphi_i}\varphi_i^{\cdot} - L(\varphi, \varphi^{\cdot}, t)\right]\right\}dt =$$

$$= \int_{t_0}^{t_1}\left\{\max_{\xi \in R^n}\left[\sum_{i=1}^n \frac{\partial W}{\partial \varphi_i}\xi_i - L(\varphi, \xi, t)\right] - \left[\sum_{i=1}^n \frac{\partial W}{\partial \varphi_1}\varphi_i^{\cdot} - L(\varphi, \varphi^{\cdot}, t)\right]\right\}dt \geqslant 0 \tag{2.7}$$

It follows from (2.3) that

$$V[q(t)] = W[q(t)] - \min_{z(t) \in I\{q\}} W[z(t)] = W[q(t)] - W[z_q(t)]$$

This means that $V[q(t)] \geqslant 0$, so $q(t) \in I\{q\}$. Therefore the value of the functional $V[q(t)] \geqslant 0$ characterizes the difference between the Hamilton action (1.6) on an arbitrary curve $q(t)$ and on the solution $z_q(t)$ of system (2.4) between the same boundary points $q(t_0) = z(t_0)$ and $q(t_1) = z(t_1)$. Using the functional (2.3) we can obtain a variational characterization of the solutions $q(t)$ of Eqs (1.1) that satisfy conditions (1.7) [5].

Consider the set

$$\Gamma = \{q(t) \in C^1[t_0, t_1],\ F_j[q(t)] \leqslant 0,\ j = 1, 2, \dots, k\} \tag{2.8}$$

of all possible functions $q(t)$ of the class $C^1[t_0, t_1]$ satisfying conditions (1.7).

Theorem 1 (*the maximum principle*). Let the Lagrangian $L(q, q^{\bullet}, t)$ be a strongly convex function of the variables $q_i^{\bullet}$ and assume that the boundary-value problem for Eqs (1.1) with conditions (1.5) has a unique solution for any q^0, q^1, $t_1 > t_0$. Then system (1.1) has a solution $\tilde{q}(t)$ that satisfies conditions (1.7) (i.e. a solution $\tilde{q}(t) \in \Gamma$) if and only if the following condition holds

$$V[\tilde{q}(t)] = \min\{V[q(t)] \mid q(t) \in \Gamma\} = 0 \tag{2.9}$$

or, what is the same, the minimax condition

$$\min_{q(t)\in\Gamma}\ \max_{z(t)\in I\{q\}} \int_{t_0}^{t_1} \{L(q(t), q^{\cdot}(t), t) - L(z(t), z^{\cdot}(t), t)\}\, dt = 0$$

Proof. Necessity. Let $\tilde{q}(t) \in \Gamma$ be a solution of Eqs (1.1) satisfying conditions (1.7). Then by the condition of the theorem $W[\tilde{q}(t)] = \min\{W[q(t)] \mid q(t) \in I[\tilde{q}(t)]\}$, whence it follows that $V[\tilde{q}(t)] = 0$. Since the functional $V[q(t)]$ is non-negative, it follows that the function $\tilde{q}(t) \in \Gamma$ makes the functional $V[q(t)]$ take its absolute minimum $V[\tilde{q}(t)] = 0$ so that condition (2.9) holds.

Sufficiency. If condition (2.9) holds, it follows by expression (2.3) for $V[q(t)]$ that

$$W[\tilde{q}(t)] = \int_{t_0}^{t_1} L(\tilde{q}, \tilde{q}^{\cdot}, t)\, dt = \min\left\{\int_{t_0}^{t_1} L(q, q^{\cdot}, t)\, dt \mid q(t) \in I\{\tilde{q}\}\right\}$$

In that case it follows from Hamilton's principle that the curve $\tilde{q}(t)$ is a solution of Lagrange's equations (1.1) [1]. Since by assumption $\tilde{q}(t) \in \Gamma$ this solution will also satisfy conditions (1.7). This completes the proof of Theorem 1.

When conditions (1.7) reduce to equalities (1.5), the functional $V[q(t)]$ differs from the Hamilton action by a constant quantity and the minimax principle directly yields Hamilton's principle.

Note that Theorem 1 remains true not only for sets Γ defined by inequalities (1.7), but also for sets defined by conditions of arbitrary form.

3. THE MAXIMUM PRINCIPLE WHEN CONJUGATE POINTS EXIST

We will now consider the general case of a mechanical system (1.1) for which the interval $[t_0, t_1]$ may contain conjugate points. In that case the principle of least action fails to hold and the Hamilton action (1.6) will have stationary values on solutions of (1.1), (1.5). Hence the functional $V[q(t)]$ cannot be defined as in (2.3). To construct an analogue of $V[q(t)]$ we shall use a special construction, which reduces to (2.3) when system (1.1) has a unique solution for Lagrange boundary conditions (1.5). It is this construction that enables us to extend the maximum principle to mechanical systems with conjugate points and to give a variational description of solutions of Lagrange's equations (1.1) with arbitrary conditions (1.7).

Theorem 2. Let the Lagrangian $L(q, q^{\bullet}, t)$ of system (1.1) be a strongly convex function of the variables $q_i^{\bullet}$. Then for any function $y(t) \in C^1[t_0, t_1]$ there is some $M > 0$ for which one can construct a system of $N \geqslant N_0\{y(t)\}$ cones G_s^M in the extended $(n+1)$-dimensional coordinate space $\{q, t\}$

$$G_s^M = \{q, t: |q_i - y_i(\tau_s)| \leqslant M(t - \tau_s),\ \tau_s \leqslant t \leqslant \tau_{s+1} \,|\tau_{s+1} - \tau_s| \leqslant (t_1 - t_2)/N_0\}$$
$$(s = 1, 2, \dots, N) \tag{3.1}$$

with apices at the points $D_s(y(\tau_s), \tau_s)$ (see Fig. 1), such that within each cone there is a central field, i.e. through any two points $\{y(\tau_i), \tau_i\}$ and $\{y(t), \tau\}, \in [\tau_i, \tau_{i+1}]$ inside one cone there is a unique solution $q(t)$ of system (1.1), which itself remains within the cone G_s^M: $\| q^{\bullet}(t) \| \leqslant M$. Any point $B \in G_j^M$ $(j = 1, 2, \dots, N)$, can be connected to the apex $D(y(\tau_j), \tau_j)$ of the cone by a unique solution $q(t)$ of system (1.1), for which moreover $\| q^{\bullet}(t) \| \leqslant M_1$, where M_1 is some number $(M < M_1 < \infty)$.

Before proving the theorem we need a lemma which establishes an important property of the Lagrange equations (1.1). If system (1.1) is solved for the highest-order derivatives, i.e. put in the form of (1.4), then the boundary-value problem for (1.4) with conditions (1.5) reduces to an integral equation

$$q(t) = \frac{at_1 - bt_0}{t_1 - t_0} + \frac{b - a}{t_1 - t_0} t + \frac{t_0 - t}{t_1 - t_0} \int_{t_0}^{t_1} d\tau \int_{t_0}^{\tau} f(q(\xi), q^{\cdot}(\xi), \xi) d\xi +$$
$$+ \int_{t_0}^{t} d\tau \int_{t_0}^{\tau} f(q(\xi), q^{\cdot}(\xi), \xi) d\xi$$

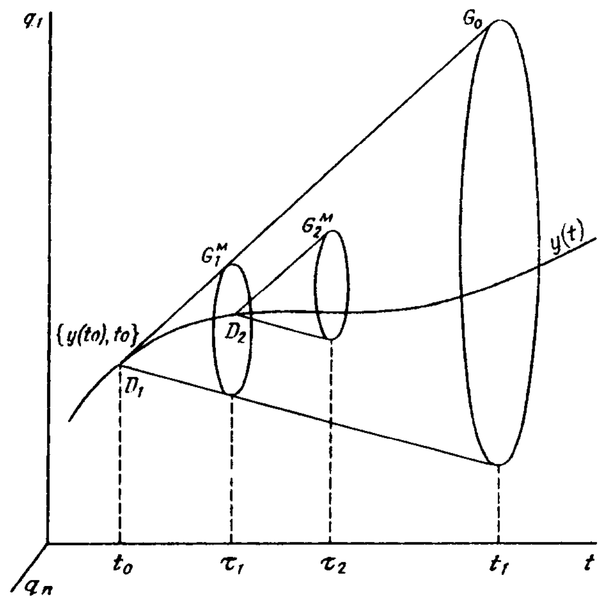

FIG. 1.

Differentiating with respect to t, we see that this equation becomes

$$q^{\cdot} = R\{q^{\cdot}(t)\} \tag{3.2}$$

$$R\{q^{\cdot}(t)\} = \frac{b-a}{t_1-t_0} - \frac{1}{t_1-t_0}\int_{t_0}^{t_1} d\tau \int_{t_0}^{\tau} f\left(a+\int_{t_0}^{\theta} q^{\cdot}(\xi)d\xi,\ q^{\cdot}(\theta),\ \theta\right)d\theta +$$

$$+\int_{t_0}^{t} f\left(a+\int_{t_0}^{\theta} q^{\cdot}(\xi)d\xi,\ q^{\cdot}(\theta),\ \theta\right)d\theta$$

The function $f(q,\ q^{\cdot},\ t)$ is defined by the right-hand side of system (1.4), i.e. by Lagrange's equations, and the constant term $(b-a)/(t_1-t_0)$ is determined by the boundary conditions (1.5). The operator $R\{q^{\cdot}(t)\}$ will be considered for different initial points $(a,\ t_0)$ and final points $(b,\ t_1)$.

Lemma 1. Under the assumptions of Theorem 2, if the distance between the points $(a,\ t_0)$, $(b,\ t_1)$ is sufficiently small, the operator $R\{q^{\cdot}(t)\}$ defined in (3.2) has a unique fixed point $q^{\cdot}(t)=R\{q^{\cdot}(t)\}$ in a sphere $\|q^{\cdot}(\cdot)\|\leqslant M$, where $M<\infty$ is some number.

The proof of Lemma 1 is based on a modification of the contracting mapping principle [6]. Choose an arbitrary number $\alpha\in(0,\ 1)$ and a number $M>0$ such that

$$M \geqslant (1-\alpha)^{-1}\max\{\|R\{0\}\|,\ \|a-b\|/(t_1-t_0)\} \tag{3.3}$$

The operator $R\{q^{\cdot}(t)\}$ satisfies the inequality

$$\|R\{q_1^{\cdot}(t)\}-R\{q_2^{\cdot}(t)\}\| \leqslant 2(t_1-t_0)\|f(q_1(\cdot),\ q_1^{\cdot}(\cdot),\cdot)-f(q_2(\cdot),\ q_2^{\cdot}(\cdot),\cdot)\|_{[t_0,t_1]},$$

$$q(t) = a+\int_{t_0}^{t} q^{\cdot}(\tau)d\tau \tag{3.4}$$

$$\|X(\cdot)\|_{[t_0,t_1]} = \max_{t\in[t_0,t_1]}\left(\sum_{i=1}^{n} X_i^2(t)\right)^{1/2}$$

The function $f(q,\ q^{\cdot},\ t)$ satisfies a Lipschitz condition in the sphere $\|q^{\cdot}(\cdot)\|\leqslant M$ with constant

$$K(M) = \max_{\substack{\|q^{\cdot}(\cdot)\|\leqslant M,\ t_1\leqslant t\leqslant t_0 \\ \|q(\cdot)\|\leqslant\|a\|+(t_1-t_0)M}}\left\{\left\|\frac{\partial f(q,q^{\cdot},t)}{\partial q}\right\|,\ \left\|\frac{\partial f(q,q^{\cdot},t)}{\partial q^{\cdot}}\right\|\right\} \tag{3.5}$$

It follows that in the interval

$$T = \tau_1-t_0 \leqslant \alpha/[4K(t_1-t_0)] \tag{3.6}$$

the boundary-value problem for system (1.1) with boundary conditions

$$q(t_0) = a,\quad q(\tau_1) = b$$

has a unique solution. Indeed, in that case, by (3.4), the operator $R\{q^{\cdot}(t)\}$ will satisfy a Lipschitz condition in the sphere $B=\{\|q^{\cdot}(t)\|\leqslant M,\ t_0\leqslant t\leqslant \tau_1\}$, with a constant $\alpha_1\leqslant\alpha<1$, i.e.

$$\|R\{q_1^{\cdot}(\cdot)\}-R\{q_2^{\cdot}(\cdot)\}\| \leqslant \alpha_1\|q_1^{\cdot}(\cdot)-q_2^{\cdot}(\cdot)\| \tag{3.7}$$

We will now show that $R\{q^{\cdot}(t)\}$ has a unique fixed point $q^{\cdot}=R\{q^{\cdot}\}$ in the sphere $B\{\|q^{\cdot}\|\leqslant M\}$.

To that end, consider the iterative sequence

$$q^{\cdot}_{p+1} = R\{q^{\cdot}_p\}, \quad q^{\cdot}_0 \equiv 0$$

By (3.3), the first term $q^{\cdot}_1$ of this sequence lies in B, i.e. $q^{\cdot}_1 \in B$. Suppose that for all $m \leqslant s$ we have $q^{\cdot}_m \in B$. It can be shown that this inclusion will also be true for $m = s+1$.

We will now show that the sequence $\{q^{\cdot}_m\}$ converges. For any $p \geqslant 1$, by (3.7)

$$\| q^{\cdot}_{m+p}(t) - q^{\cdot}_m(t) \| \leqslant \alpha \| q^{\cdot}_{m+p-1}(t) - q^{\cdot}_{m-1}(t) \| \leqslant \alpha^m \| q^{\cdot}_p(t) \| \leqslant \alpha^m M \to 0, \quad m \to \infty$$

Hence $\{q^{\cdot}_m\}$ is a Cauchy sequence. It therefore converges to a limit in the complete metric space of continuous functions with the uniform norm, and this limit is a fixed point of the operator $R\{q^{\cdot}(t)\}$.

Since $\lim_{m\to\infty} q^{\cdot}_m(t) = q^{\cdot}(t)$ and $\| q^{\cdot}_m(t) \| \leqslant M$, $m = 0, 1, 2, \ldots$, and the convergence is uniform, $\| q^{\cdot}(t) \| \leqslant M$, that is $q^{\cdot}(t) \in B\{\| q^{\cdot} \| \leqslant M\}$.

That the fixed point is unique may be proved indirectly, using the Lipschitz condition (3.7) and the inequality $0 < \alpha < 1$.

Proof of Theorem 2. We will show that, under the assumptions of the theorem, one can construct a system of cones to cover an arbitrary curve $y(t) \in C^1[t_0, t_1]$, in such a way that the principle of least action will hold in each cone, i.e. within each cone a central field of extremals exists.

Let $\beta = \max_{t\in[t_0,t_1]} \| y^{\cdot}(t) \|$. For some number $\alpha \in (0, 1)$, choose a number $M \geqslant \beta$ satisfying inequality (3.3)

$$M \geqslant (1-\alpha)^{-1} \max\{\| R\{0\} \| \mid a = y(t_0),\ b = y(t_1)\}$$

In the extended coordinate space $\{q, t\}$, construct an auxiliary cone G_0 which completely covers the curve $y(t)$ and is such that

$$G_0 = \{q, t\colon\ t \in [t_0, t_1],\ | q_i(t) - a | \leqslant M(t - t_0)\} \tag{3.8}$$

Define a constant

$$M_1 \geqslant (1-\alpha)^{-1} \max\{R\{0\} \mid a(t_0) = y(t_0),\ (b, t_1) \in G_0\}$$

By Lemma 1, if $[t_0, \tau_1]$ is an interval satisfying the estimate (3.6) for $K = K(M_1)$, we can construct a cone (see Fig. 1)

$$G_1^M = \{q, t\colon\ t \in [t_0, \tau_1],\ | q_i(t) - a | \leqslant M(t - t_0)\} \tag{3.9}$$

in which there is a central field of extremals. The apex $D_1(y(t_0), t_0)$ of G_1^M can be connected to any point $\{b(\tau), \tau\} \in G_1^M$ $(\tau > t_0)$ by a unique solution $z(t)$ of the Lagrange system (2.4) or of the corresponding operator equation $z^{\cdot}(t) = R\{z^{\cdot}(t)\}$. This function $z(t)$ will have a uniformly bounded velocity

$$\| z^{\cdot}(t) \|_{[t_0, \tau_1]} \leqslant M_1$$

If $B(b(\tau), \tau)$ is a point on the curve $y(t)$, the corresponding solution $z(t)$ of system (2.4), connecting it with the apex $(y(t_0), t_0)$, will satisfy the condition $\| z^{\cdot}(t) \|_{[t_0, \tau_1]} \leqslant M$. As with (3.9), using Lemma 1, we can construct a cone G_2^M with apex at $D_1(y(\tau_1), \tau_1)$. We thus obtain N_0 cones covering the curve $y(t)$, where $N_0 = N_0[y(t)] = V[(t_1 - t_0)/T]$, the value of T is determined by (3.6). This completes the proof of Theorem 2.

In the cones G_s^M $(s = 1, 2, \ldots, N)$ that form a cover of the curve $y = y(t)$, there is a central field of extremals of the functional (1.6) [3] (with centres in D_s), and so the principal Hamiltonian function $W[y(\tau_s), \tau_s, q, t]$ is defined. Hence the principle of least action holds in each cone G_s^M

$$W[q^s(t)] = \min\{W[q(t)] \mid q(t) \in I_s\{y(t)\}\},$$

where the set of paths $I_s\{y(t)\}$ is defined by

$$I_s\{y(t)\} = \{q(t) \in C^1\colon\ q(\tau_{s-1}) = y(\tau_{s-1}) \tag{3.10}$$

$$q(\tau_s) = y(\tau_s),\ q(t) \in G_s^M\}, \quad s = 1, 2, \ldots, N_0$$

Everything is now ready to construct a functional that will henceforth play the role of (2.3) for systems with conjugate points.

For any function $q(t) \in C^1$ we define a non-negative functional

$$V^*[q(t)] = \sum_{s=1}^{N} \max\left\{ \int_{\tau_{s-1}}^{\tau_s} \{L(q, q^\cdot, t) - L(z, z^\cdot, t)\}\, dt \ \middle|\ z(t) \in I_s\{q\} \right\} \tag{3.11}$$

where $N \geq N_0[q(t)]$ is an integer. For this functional we can prove an analogue of Theorem 1.

Theorem 3. Let the Lagrangian $L(q, q^\cdot, t)$ of system (1.1) be strongly convex with respect to $q_i^\cdot$. Then a solution $\tilde{q}(t) \in C^1[t_0, t_1]$ of system (1.1) satisfying conditions (1.7) will exist if and only if the following minimax condition holds

$$V^*[\tilde{q}(t)] = \min\{V^*[q(t)] \mid q(t) \in \Gamma\} = 0 \tag{3.12}$$

where

$$N = \max\{N_0[q(t)]\} \tag{3.13}$$

or, written out in full

$$\min_{q(t) \in \Gamma} \sum_{s=1}^{N} \max_{z^s(t) \in I_s} \left[\int_{t_{s-1}}^{\tau_s} L(q(t), q^\cdot(t), t)\, dt - \int_{\tau_{s-1}}^{\tau_s} L(z^s(t), z^{\cdot s}(t), t)\, dt \right] = 0$$

The number $N_0[z(t)]$ in (3.13) was defined in Theorem 2; the set I_s is defined by (3.10) for each cone G_s^M $(s = 1, 2, \ldots, N)$ in the set of cones defined for any curve $y(t) \in C^1$ by (3.1).

Proof of Theorem 3. The proof follows the same lines as that of Theorem 1, using Theorem 2.

Thus, any solution $\tilde{q}(t)$ of system (1.1) that satisfies conditions (1.7) may be defined by the relation

$$V^*[\tilde{q}(t)] = 0, \quad \tilde{q}(t) \in \Gamma$$

This means that all these solutions lie at the intersection of the set of functions Γ and the set of zeros of the functional V^* defined in (3.11). If system (1.1) has no conjugate points, then V^* will be identical with the functional V defined in (2.3).

The minimax condition (3.12) enables one to solve boundary-value problems for system (1.1) by first applying direct variational methods and then solving a finite-dimensional minimax problem.

Sometimes the domains G_s^M in which there are no conjugate points for system (1.1) need not be cones in the extended phase space.

For example, if the Lagrangian of the system is

$$L(q, q^{\cdot}, t) = \sum_{i,k=1}^{n} \{a_{ik}(t) q_i^{\cdot} q_{ik}^{\cdot} + b_i(q,t) q_i^{\cdot}\} + C(q,t)$$

then the right-hand side of Eqs (1.4), solved for the highest-order derivatives

$$q_i^{\cdot\cdot} = \sum_{s=1}^{N} d_{is}(t) q_s^{\cdot} + g_i(q,t)$$

will be a linear function of the variables $q_s^{\cdot}$ ($s = 1, 2, \ldots, N$). Hence the Lipschitz constant $K(M)$ in (3.5) depends only on the restrictions on the domain $G\{q, t\}$ in the extended coordinate space and not on $q^{\cdot}$.

For a harmonic oscillator with Lagrangian

$$L(q, q^{\cdot}, t) = \tfrac{1}{2}(q^{\cdot 2} - \omega^2 q^2)$$

the right-hand side of Eqs (1.4) is linear in q and $q^{\cdot}$. Therefore the domains $G_s = \{q, t: \tau_{s-1} \leqslant t \leqslant \tau_s, \tau_s - \tau_{s-1} < \pi/(2\omega)\}$ may be defined as the strips in $\{q, t\}$ space bounded by hyperplanes $\tau_s = \text{const}$, $s = 1, 2, \ldots, N$ with interval $\pi/(2\omega)$.

It has been proved [7] that a solution of the boundary-value problem consisting of the system $q_i^{\cdot\cdot} = F_i(q, t)$ and conditions (1.5) exists and is unique for $0 < t_1 \leqslant (\sqrt{(2K)}(n+1))^{-1}$, where $K = \max_{i,k,q} \|\partial F_i/\partial q_k\|$. Theorem 3 yields an analogous estimate $0 < t_1 \leqslant (2\sqrt{(K(M))})^{-1}$, with the constant $K(M)$ as defined in (3.5).

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Translated by D.L.